

# Non-Abelian Geometry

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## Abstract

Spatial noncommutativity is similar and can even be related to the non-Abelian nature of multiple D-branes. But they have so far seemed independent of each other. Reflecting this decoupling, the algebra of matrix valued fields on noncommutative space is thought to be the simple tensor product of constant matrix algebra and the Moyal-Weyl deformation. We propose scenarios in which the two become intertwined and inseparable. Therefore the usual separation of ordinary or noncommutative space from the internal discrete space responsible for non-Abelian symmetry is really the exceptional case of a unified structure. We call it *non-Abelian geometry*. This general structure emerges when multiple D-branes are configured suitably in a flat but varying  $B$  field background, or in the presence of non-Abelian gauge field background. It can also occur in connection with Taub-NUT geometry. We compute the deformed product of matrix valued functions using the lattice string quantum mechanical model developed earlier. The result is a new type of associative algebra defining non-Abelian geometry. Possible supergravity dual is also discussed.

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## 1. Introduction

This paper is devoted to the search and study of certain unusual and hitherto unknown facets of noncommutative space from string and field theories and quantum mechanics. Introducing noncommutativity as a way of perturbing a known field theory has received much interest recently (see [1,2,3] and the references therein), and hence we shall refrain from repeating the usual motivations and excuses for doing it. Another reason lies in string theory itself. The antisymmetric tensor field  $B$  in the Neveu-Schwarz – Neveu-Schwarz sector of string theory, while simpler than its cousins in the Ramond – Ramond sector, is still shrouded in mystery and surprisingly resistant to an unified understanding. One of its many features is its relation to spatial noncommutativity. Let us recall it briefly.

Open strings interact by joining and splitting. This lends naturally to the picture of a geometrical product of open string wave functionals that is clearly noncommutative. One may formulate a field theory of open strings based on this noncommutative product the same way as conventional field theory is formulated on products of wave function fields[4]. But the string wave functional is unwieldy and its product is enormously complex. Noncommutativity certainly does not help. To learn more we have to do with less. One way is to truncate the theory to a low energy effective theory of the small set of massless fields. Another is to approximate the string by a minimal “lattice” of two points. This is especially well suited to mimicking the geometric product of open strings. It emerges from both approximations that, at least using some choice of variables, the natural product of wave function fields is the following noncommutative deformation of the usual one:

$$(\Psi * \Phi)(x) = \exp \left( \frac{i}{2} \frac{\partial}{\partial x'^{\mu}} \Omega^{\mu\nu} \frac{\partial}{\partial x''^{\nu}} \right) \Psi(x') \Phi(x'') \Bigg|_{x'=x''=x}. \quad (1.1)$$

The parameter of noncommutativity  $\Omega$  is expressed in terms of the spacetime metric<sup>1</sup>  $G$  and  $B$  by

$$\Omega = -(2\pi\alpha')^2 G^{-1} B G^{-1} (1 - (2\pi\alpha')^2 B G^{-1} B G^{-1})^{-1}. \quad (1.2)$$

It should be noted that noncommutativity is not a consequence of  $B$  being nonvanishing or large. It is intrinsic to the geometry of smooth string junctions that a canonical product exists for the functions on the space of open paths in the target space manifold with the appropriate boundary condition, and that product is noncommutative. The approximations mentioned above induces noncommutativity in the algebra of functions on the submanifold

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<sup>1</sup> Note that  $G$  in this paper and in [5] is the same as the “closed string metric”  $g$  in [3], and the noncommutativity parameter  $\Omega$  here is the same as  $\Theta$  there.

to which the end points are restricted, namely the D-brane. The algebra actually becomes commutative in the limit of very large  $B$ !

It is a glaring deficiency of the present understanding from string theory that one knew only how to deal with constant and flat  $B$  field. Introducing curvature for  $B$  takes string theory away from the usual sigma model to a rather different realm, so understanding it fully seems to call for drastic conceptual advance. On the other hand, varying but flat  $B$  field should be accessible by the available technology but is hampered by technical difficulties. For example, a formal construction of a noncommutative product using an arbitrary Poisson structure in place of the constant  $\Omega$  has been given by Kontsevich[6]. The construction made essential use of a degenerate limit of sigma model [7]. But the result employs some very abstruse mathematics and its convergence properties essentially unknown. Behind the complication must lie some interesting and novel structures that needs to be deciphered.

We propose, as a first step toward understanding such situation from string theory, probing it with multiple parallel D-branes configured in a way such that each D-brane only senses a constant but respectively different  $B$  field. On each D-brane the usual noncommutative algebra incorporates the effect of the locally constant  $B$  field without knowing that  $B$  is actually varying. The latter is revealed in the communication among different D-branes via the open strings that start and end on different D-branes. We study the wave functions associated with such “cross” strings and find that their product is deformed in a new and intriguing way that retains associativity. Along the same line of the reasoning as in [3], one expects that it is in terms of this product that the effect of  $B$  field is best described at the zero slope limit. As D-branes are dynamical objects inclined to fluctuate, this picture is necessarily an idealization, describing the limit where the effect of such fluctuation is very small. It would be worthwhile to study quantitatively the corrections due to such fluctuation.

One can better appreciate the import of this new deformation of algebra by recalling another player. It is an essential feature of a D-brane that it has a gauge symmetry and an associated gauge connection. Let us briefly recollect some of the well known facts relevant here. In the simplest and most common circumstances, a single D-brane has an  $U(1)$  symmetry, and a multiplet of  $N$  D-branes on top of each other collectively have an  $U(N)$  symmetry, with fields that are  $N \times N$  matrices transforming in the adjoint representation of this  $U(N)$ . When the algebra of functions on the D-brane submanifold is deformed, so is the gauge symmetry. For  $U(1)$ , the new Lie algebra is given by the commutator of the deformed product and is no longer trivial. For  $U(N)$ , the product of the matrix valued functions becomes

$$(M * N)_k^i(x) = \sum_j \left( M_j^i * N_k^j \right) (x). \quad (1.3)$$

And the deformed Lie algebra originates from the commutator of this new matrix algebra. In this deformation the noncommutativity of spacetime and the non-Abelian property of multiple D-branes are simply and independently tensored together and do not affect each other, yet.

It has long been known that the  $U(1)$  (trace) part of the field strength  $F$  always appear together with  $B$  as  $(B - F_{U(1)})$ . A certain gauge symmetry actually connects the two. Therefore  $F$  also contributes to noncommutativity and appears in the expression for  $\Omega$  by replacing  $B$  with  $(B - \text{tr } F/N)$ . What about the non-Abelian part of  $F$ ? Let us consider a constant background for  $F$ , as varying  $F$  is again too difficult. For this constraint to make sense it has to be  $U(N)$  covariant, i.e. it should be covariantly constant. Hence we also choose  $F$  so that different spatial components of  $F$  are in some Cartan subalgebra of  $N \times N$  matrices. By a choice of basis we can make them all diagonal. Such background generically breaks  $U(N)$  down to  $U(1)^N$  and it is meaningful to talk about  $N$  distinct branes, each with a constant field strength of its own unbroken  $U(1)$ . This poses the same problem as the early configuration of multiple D-branes probing transversally varying  $B$  field but with a different interpretation<sup>2</sup>. Now we consider a intrinsically non-Abelian deformation of the matrix product on the D-branes. As it turns out, this new product is no longer the simple tensoring of the star product (eq. 1.1) and the usual matrix algebra. The noncommutative “real” space and the non-Abelian internal space mix and become inseparable. We call this *non-Abelian geometry*, and give a general formulation and the underlying philosophy at the end of section 3.

In this work, we have found a large class of examples of this new geometry by considering non-trivial D-branes configurations with non-Abelian field content and/or under the influence of non-flat  $B$  field. The concrete form of the product are derived from a lattice approximation of string theory in section 2 and 3, and presented here. There are different ways to express the product, corresponding to different choices of operator ordering. With the “symmetric” ordering defined and used throughout section 2, the geometry is defined by an algebra with the following product.

$$(\Psi * \Phi)_j^i(x) \equiv \sum_l \exp \left( \frac{i}{2} \frac{\partial}{\partial x'^{\mu}} \Omega_{il;lk}^{\mu\nu} \frac{\partial}{\partial x''^{\nu}} \right) \Psi_l^i(x') \Phi_k^l(x'') \Bigg|_{x''=S_{jk}^{ik}x}^{x'=S_{ij}^{ik}x}. \quad (1.4)$$

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<sup>2</sup> One difference is that in the configuration with varying flat  $B$  field, an open string stretching between two D-branes would have a mass offset proportional to the separation between them. It's possible to take a special limit for the components of the close string metric along the separation of the D-branes to make the offset vanish. However, here we are only studying the kinematics encoded in the algebra connecting all the  $(i, j)$  strings and this offset is irrelevant.

with  $S$  and  $\Omega$  satisfying (section 2.2)

$$\begin{aligned} S_{j_1 j_2}^{j_3 j_4} &= S_{j_1 j_2}^{k_1 k_2} S_{k_1 k_2}^{j_3 j_4}, \\ S_{j_1 j_2}^{j_3 j_4} \Omega_{j_3 j_4; j_5 j_6} &= \Omega_{j_1 j_2; j_5 j_6} = \Omega_{j_1 j_2; j_3 j_4} S_{j_5 j_6}^{j_3 j_4 \top}. \end{aligned} \quad (1.5)$$

Here no summation over Latin (Yang-Mills) indices takes place, but summation over repeated suppressed Greek (space-time) indices does take place. With the split ordering introduced in section 3, the product takes the form

$$(\Psi \times \Phi)_j^i(e^a, e^A) \equiv \sum_{k, l} \Psi_k^i(e^a, M^A) \exp \left( i \frac{\partial}{\partial M^A} \hat{\Omega}^{Aa} \frac{\partial}{\partial M^a} \right)_{kl} \Phi_j^l(M^a, e^A) \Bigg|_{M^A=0=M^a}, \quad (1.6)$$

Here  $\hat{\Omega}^{Aa}$  is a matrix with Latin indices  $k$  and  $l$  the exponential is the usual exponential of a matrix. In this paper we shall derive (eq. 1.4) and (eq. 1.6) from certain physical backgrounds in string theory, so they appear in concrete and specific context with motivation from string theory. However, we should emphasize here that with the forms of the products now known, we can and should consider them independently and abstractly as examples of non-Abelian geometry, and look for their appearance in other contexts as well. In deriving (eq. 1.4) the parameters  $S$  and  $\Omega$  takes on values determined by the physical background in our setup. However, (eq. 1.4) stands as a valid definition of an associative product as long as (eq. 1.5) holds. Similarly, we have derived (eq. 1.6) in section 3 for the case of  $SU(2)$ , but these forms of product apply more generally for arbitrary  $\hat{\Omega}_{kl}^{Aa}$ . Unlike the symmetric ordering, here  $\hat{\Omega}$  is already “gauge fixed” and associativity imposes no constraint on them. The two form should be related to each other by a change of ordering. This is clear in the examples discussed in this paper, though we have not worked out the general and explicit transformation relating the two in this paper. Finally we note that even in such general forms, they only represent a certain class of non-Abelian geometry. The explicit forms of non-Abelian geometry in its full generality is a very interesting problem still under investigation.

At this point one may well consider other approaches to generalizing D-brane geometry. One very interesting approach in recent time has been the efforts to study the geometry of D-branes with vector bundles in Calabi-Yau manifolds ([8] and the references therein). There one takes the D-brane wrapping supersymmetric cycles in Calabi-Yau manifolds and the vector bundle on the D-branes as a whole and study their properties in relation to target space supersymmetry and mirror symmetry. It would be very relevant to fully reconcile these two facets of D-branes: the vector bundle aspects of D-branes steeped in conventional commutative geometry as one, and the non-commutative geometry that the

open string fields see as the other. However, non-commutativity introduces such tremendous technical challenges in curve space that novel and powerful methods and concepts seems necessary to tackle it. This paper provides one possibility in dealing with curvature in the antisymmetric tensor  $B$  field (section 2.2). It might prove helpful in dealing with curvature in the metric through various correspondences such as T-duality, which relates the metric and the  $B$ -field.

Here is an outline of the paper. The non-Abelian noncommutative product is explicitly constructed in section two. It turns out a two point lattice approximation to quantum mechanics is perfectly suited for this purpose. A systematic methodology of computing product was developed in [5]. We review and elaborate it in section 2.1. In section 2.2 we apply it to the most general case of non-Abelian noncommutativity and obtain the main result of the paper (eq. 2.55). In section 3 we then turn to the specific case of the deformation parameter being in the adjoint of  $SU(2)$  and use a variation of the method presented in section two. The result is a surprisingly compact and highly suggestive form of the new product (eq. 3.16). The situation of multiple noncommutativity parameters also makes an appearance in connection with Taub-NUT geometry<sup>3</sup>. We shall discuss in section four how a whole class of Lorentz non-invariant theories governed by the  $B$  field dynamics can be studied in an unified way. We conclude with a discussion on the possible gravity dual for the system we study as well as some other related issues.

## 2. Construction of the non-Abelian noncommutative product

### 2.1. Review and elaboration

#### *The origin of noncommutativity*

A classic and salient feature of string theory is its geometric appeal. For example, strings interact by smoothly joining and splitting. In conventional field theories, one can visualize an interaction of particles as a vertex of intersection by propagators in a Feynman diagram. The well known rule from perturbation theory states that each term in the interaction Lagrangian gives rise to a distinct kind of such vertex. Interaction at a point corresponds to product of fields at the same point. The rule of string theory perturbation is entirely analogous. However, the algebra of the product, besides being obviously much more complicated, has a new twist. Consider the joining of two or more

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<sup>3</sup> For branes near a conifold, non-Abelian noncommutativity makes an appearance in the fractional brane setup [9].

open strings into one. It should be apparent that this process is *not* commutative though still clearly associative. The multiplication between the wave functionals of the open string, also known as the open string fields, share the same property<sup>4</sup>.

Intuitively, the product seems easy to define. Let  $\Psi[\gamma]$  and  $\Phi[\gamma]$  be two string wave functionals, where the argument  $\gamma$  is an open path in the target space with the proper boundary conditions. The geometric product defined above can be written as

$$(\Psi \cup \Phi)[\gamma] = \int D[\gamma_1] D[\gamma_2] \delta[\gamma = \gamma_1 \uplus \gamma_2] \Psi[\gamma_1] \Phi[\gamma_2], \quad (2.1)$$

The operation  $\uplus$  is just the geometric process of “joining” defined above, with a refining sensitivity to sign and orientation so that a segment that backtracks itself also erases itself. This “definition” manifests noncommutativity<sup>5</sup> and associativity, but it is also horribly divergent and ill defined. One can remedy this with an elaborate procedure [4] but there is an alternative way to make sense of this product, if one is willing to forgo the bulk of the data encoded in the string field in exchange for a better understanding of what remains.

Before we do this first recall that the standard string action is

$$\begin{aligned} S = & \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \left[ G_{\mu\nu} \left( \dot{X}^{\mu} \dot{X}^{\nu} - X'^{\mu} X'^{\nu} \right) \right] \\ & + \int_{\partial_2 \Sigma} d\tau \mathcal{A}_{\mu}(X) \dot{X}^{\mu} - \int_{\partial_1 \Sigma} d\tau \mathcal{A}_{\mu}(X) \dot{X}^{\mu}. \end{aligned} \quad (2.2)$$

Here the subscripts “2” and “1” on  $\partial\Sigma$  label the “left” and “right” boundaries of the open string worldsheet.  $G$  is the background closed string metric and assumed to be constant. Usually there would also be a term  $\frac{1}{4\pi\alpha'} B_{\mu\nu} \left( \dot{X}^{\mu} X'^{\nu} - X'^{\mu} \dot{X}^{\nu} \right)$  in the action. However we would only be dealing with flat  $B$  field, and in  $R^D$  flat  $B$  is exact and equal to  $dA'$  for some  $A'$ . We henceforth include  $-A'$  implicitly in  $\mathcal{A}$  so that  $d\mathcal{A} = F - B \equiv \mathcal{F}$ .

Let us be careful with boundary conditions from now on. To solve the equation of motion we need to impose one for each boundary component. We want the two ends of the strings to move only within two possibly distinct but parallel D-branes of the same dimensions. For the purpose at hand, we will only be concerned with coordinates that parameterize the D-branes’ worldvolume under the influence of a nondegenerate  $\mathcal{F}$  and ignore from now on all the other coordinates, including those along which the D-branes

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<sup>4</sup> This observation was made clear in [4], where one can also find relevant graphic illustrations.

<sup>5</sup> Even though we have made no mention of  $B$ !

separate. We shall only consider situations for which this space is  $R^D$ . The boundary condition for the relevant coordinate fields is

$$\frac{1}{2\pi\alpha'} G_{\mu\nu} X'^{\nu} = \mathcal{F}_{\mu\nu} \dot{X}^{\nu}. \quad (2.3)$$

For the problem to be tractable using the method in this paper, we also require  $\mathcal{F}$  to be constant. Note that since  $\mathcal{F}$  is evaluated only at the ends of the strings, confined to the D-branes, this requirement only enforces the constancy of the pull-back of  $B$  to the D-branes.  $B$  may vary in directions transverse to the D-branes, or have components not entirely parallel to the D-branes that vary. Indeed the flatness of  $B$  correlate the last two kinds of variations. In this subsection let us consider the case of a single D-brane, so there is only one constant  $\mathcal{F}$ . We will return to the general case in the next subsection.

Now we approximate the spatial extent of the open string by the coarsest “lattice” of two points, namely the two ends, labeled by 1 and 2. Let the width of the string be  $2/\omega$ . The action (eq. 2.2) is approximated by<sup>6</sup>

$$\begin{aligned} S = \int d\tau \left[ \frac{1}{4\pi\omega\alpha'} \left( \dot{X}_1^2 + \dot{X}_2^2 - \frac{\omega^2}{2} (X_2 - X_1)^2 \right) \right] \\ + \int d\tau \left( A_{\mu}(X_2) \dot{X}_2^{\mu} - A_{\mu}(X_1) \dot{X}_1^{\mu} \right). \end{aligned} \quad (2.4)$$

We shall call this system *lattice string quantum mechanics* (LSQM). The boundary conditions now become[5]

$$\begin{aligned} D_{\mu}^1 &\equiv [G(X_1 - X_2)]_{\mu} + \frac{4\pi\alpha'}{\omega} [\mathcal{F} \dot{X}_1]_{\mu} \sim 0; \\ D_{\mu}^2 &\equiv [G(X_2 - X_1)]_{\mu} - \frac{4\pi\alpha'}{\omega} [\mathcal{F} \dot{X}_2]_{\mu} \sim 0. \end{aligned} \quad (2.5)$$

The result of canonical quantization with constraints is <sup>7</sup>

$$\begin{aligned} [X_2^{\mu}, X_2^{\nu}] &= -\imath(2\pi\alpha')^2 \left[ G^{-1} \mathcal{F} G^{-1} \left( 1 - (2\pi\alpha')^2 \mathcal{F} G^{-1} \mathcal{F} G^{-1} \right)^{-1} \right]^{\mu\nu} \\ &\equiv \imath \Omega^{\mu\nu} = -[X_1^{\mu}, X_1^{\nu}]; \\ [X_1^{\mu}, X_2^{\nu}] &= 0. \end{aligned} \quad (2.6)$$

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<sup>6</sup> A similar but different model appeared in [10].

<sup>7</sup> In comparing the results summarized here with [5] one should note that  $D^i$  defined here is equal to  $G_{\mu\nu} C_i^{\nu}$  in [5], and that there is a typographical error of a missing  $(-1)$  in front the expression on the third line in (eq. 2.12) of [5], and another  $(-1)$  on the exponent of the second parenthesis in the same expression.



These are precisely the commutation relations for the ends of the string found in [11].

### *Matrix, Chan-Paton Factor, and Noncommutative Product*

We are now only one step from the deformed product (eq. 1.1) . It is here that the LSQM approach distinguishes itself for its conceptual and technical advantage. Now that the entire continuum of the open string is distilled down to two points, the above mentioned  $\cup$  joining of two oriented paths into one reduces to the merging of two ordered pairs of points, with the second (end) of the first pair coinciding with and “cancelling” the first (start) of the second pair:  $(x_1, m) \uplus (m, x_2) = (x_1, x_2)$ . This induces a product  $*$  of two wave functions of the lattice string, entirely analogous to (eq. 2.1) :

$$(\Psi * \Phi)(x_1, x_2) = \int dm_1 dm_2 \delta(m_1 - m_2) \Psi(x_1, m_2) \Phi(m_1, x_2). \quad (2.7)$$

If this seems reminiscent of ordinary matrix product, it is no illusion. One can think of an index on a matrix as a coordinate parameterizing some discrete space<sup>8</sup>. Since a matrix carries two indices it is the wave function of a lattice string moving on this discrete space and its  $*$  product would simply be the standard matrix multiplication. Distinguishing between the contravariant and covariant indices corresponding to distinguishing the two ends of the (lattice) string by a choice of orientation. On the other hand, attaching discrete indices to string ends is none other than introducing Chan-Paton factors. In this light, the noncommutativity of open string field and of non-Abelian gauge symmetry are not just similar in their failure to commute but have a shared geometric origin and interpretation!

Now we return to *LSQM* (lattice string quantum mechanics). Its salient feature, reviewed shortly, is the truncation of the noncommutative string field algebra down to a closed noncommutative algebra of (wave) functions on the target space. The latter is something much simpler and easier to study than the full open string algebra and still carries nontrivial information, especially the effects of the  $B$  field. The known noncommutative algebra found this way is a deformation of the “classical” commutative algebra of functions. It modifies the  $U(1)$  gauge symmetry of a single D-brane experiencing this  $B$  field into a deformed one corresponding to the group of unitary transformations in a certain Hilbert space. When multiple D-branes are present so that  $U(1)$  is replaced by the non-Abelian  $U(N)$ , the  $U(N)$  group as well as the  $N \times N$  matrix algebra is also modified. The new algebra is just the tensor product of the matrix algebra and the deformed

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<sup>8</sup> The precise connection is the discrete “fuzzy” torus explained later in footnote (13) in section 3.

noncommutative algebra of the scalar functions. No essential difference in the noncommutativity of the space is introduced by having non-Abelian gauge symmetry. This then begs the question: is there some other deformation in which the discrete internal space can become fully entangled with the (noncommutative) “real” space so that it is impossible to separate them. The answer, we shall propose, is *yes*. The condition, we shall show, is that the background parameter for noncommutativity is in an appropriate sense *non-Abelian*. This can be due to a non- $U(1)$  background for the gauge field or a varying flat  $B$  configured in the manner prescribed above.

### *Defining the product*

The (eq. 2.7) almost entirely defines the rule for making product. We still have the trivial freedom of changing the overall normalization by a constant factor, which we will fix later. Yet that equation would seem to be applicable to functions on the square of  $R^D$  rather than  $R^D$  itself. Fortunately, (eq. 2.6) says that although we start from  $2D$  canonical coordinates in the LSQM, constraint (eq. 2.5) reduces the size of a complete set of commuting observables to only  $D$ , the right number for a wave function to be defined on  $R^D$  itself. At each of the two ends, there are only  $D/2$  commuting observables. Let us make some choice and call them  $E_1^a$  and  $E_2^a$ ,  $a = 1 \dots D/2$ , where the subscript labels boundary components. Together they form a complete set of commuting observables. We shall call it the “ $aa$ ” representation which diagonalizes simultaneously  $E_1^a$  and  $E_2^a$  with eigenvalues  $e_1^a$  and  $e_2^a$  respectively. For wave functions  $\Psi_{aa}(e_1^a, e_2^a)$  and  $\Phi_{aa}(e_1^a, e_2^a)$  in this representation, the adaptation of (eq. 2.7) is immediate and obvious:

$$(\Psi * \Phi)_{aa}(e_1^a, e_2^a) \propto \int dm_1^a dm_2^a \delta(m_1 - m_2) \Psi_{aa}(e_1^a, m_2^a) \Phi_{aa}(m_1^a, e_2^a). \quad (2.8)$$

The proportionality sign here signifies that we have yet to specify the overall normalization, which scales the right hand side of (eq. 2.8) by a constant factor. We will fix it later by relating to the usual commutative product.

This product is natural also from the point of view of the LSQM.  $X_1$  and  $X_2$  commute with each other. Therefore the left and right ends decouple and the Hilbert space for the LSQM is the tensor product of the Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  of two ends respectively<sup>9</sup>. Furthermore, the operator algebra in  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are generated by the same set of observables  $X^\mu$ , but from (eq. 2.6) their commutator is exactly opposite in sign. This canonically

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<sup>9</sup> Subtlety might arise for other topology but at least for  $R^D$  this factorization holds.

correlates them as complex conjugate pair of representations of the same operator algebra. To see this, choose a basis of  $R^D$  so that  $\Omega$  is brought to the canonical form:

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \quad (2.9)$$

Let  $a, b, \dots$  enumerate the first  $D/2$  coordinates and  $A, B, \dots$  the rest. Then (eq. 2.9) can be written more compactly as:

$$\begin{aligned} J^{aA} &= \delta^{a+\frac{D}{2}, A} = -J^{Aa}, \\ J^{ab} &= 0 = J^{AB}. \end{aligned} \quad (2.10)$$

In the  $aa$  representation the  $E^a$ 's are simultaneously diagonalized while the  $E^A$ 's are implemented as differentiations:

$$\begin{aligned} E_1^A &= -\imath J^{aA} \left( \frac{\partial}{\partial e_1^a} + \imath \frac{\partial}{\partial a} \alpha_1(e_1^a) \right), \\ E_2^A &= \imath J^{aA} \left( \frac{\partial}{\partial e_2^a} + \imath \frac{\partial}{\partial a} \alpha_2(e_2^a) \right). \end{aligned} \quad (2.11)$$

Here  $\alpha_1$  and  $\alpha_2$  are just the usual phase ambiguity in canonical quantization. We can naturally identify  $e_1$  and  $e_2$  by identifying the wave functions in  $\mathcal{H}_2$  to  $\mathcal{H}_1$  after complex conjugation, and requiring  $\alpha_1 = -\alpha_2$ . Thus a ket in  $\mathcal{H}_2$  is a bra in  $\mathcal{H}_1$  and vice versa. The product (eq. 2.8) can be rewritten as

$$(|\alpha\rangle \otimes \langle\beta|) * (|\theta\rangle \otimes \langle\rho|) \propto (\langle\beta||\theta\rangle)(|\alpha\rangle \otimes \langle\rho|). \quad (2.12)$$

Although this product has manifest noncommutativity and associativity, the wave functions are not functions on the target space and there is no parameter visible that controls the noncommutativity. This is a fitting time to remember that an associative algebra has both additive and multiplicative structures but (eq. 2.8) defines only the latter. We want our algebra to be a deformation, in its multiplicative structure, of the algebra of functions on  $R^D$ , so it should be identified with the set of functions on  $R^D$  as a vector space. Wave functions in the  $aa$  representation clearly does not suit this purpose. We need to find an “ $aA$ ” representation in which a set of  $D$  observables that can pass as coordinates on  $R^D$  are simultaneously diagonalized. That is tantamount to requiring the action of translation generator  $P$  on them should be what is expected of  $R^D$  coordinates. We call them *geometric observables*.

For example, in the LSQM above  $P = \Omega^{-1}(X_1 - X_2)$ . A particularly symmetric choice for the geometric observables is simply the center of mass coordinates  $X_c^\mu$  of the lattice string system:

$$X_c^\mu = \frac{1}{2}(X_1^\mu + X_2^\mu) \quad (2.13)$$

such that

$$[X_c^\mu, P_\nu] = i\delta_\nu^\mu. \quad (2.14)$$

We can rewrite the product (eq. 2.8) in terms of functions of the eigenvalue  $x$  of  $X_c$  by using the change of basis function<sup>10</sup>

$$\langle x | e_1^a, e_2^a \rangle = \delta \left( x^a - \frac{1}{2}(e_1 + e_2)^a \right) \exp \left( -\frac{i}{4} x^A J_{Aa} (e_2 - e_1)^a \right) \quad (2.15)$$

and find that, in terms of  $\Omega$ , (eq. 2.8) is explicitly given by

$$(\Psi * \Phi)(x) = \exp \left( \frac{i}{2} \Omega^{\mu\nu} \frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x''^\nu} \right) \Psi(x') \Phi(x'') \Big|_{x'=x''=x}. \quad (2.16)$$

Here we have fixed the overall normalization mentioned before by requiring it to reproduce the usual commutative product when  $\Omega = 0$ .

## 2.2. Non-Abelian Deformation

Now we come to the main task of this paper and consider the possibility of more than one noncommutativity parameter. For each of such parameter we can define the  $*$  product above and have a distinct algebra. Let us assign labels ranging from 1 to  $N$  to this group of parameters  $\Omega_i$ . We denote elements of the  $i$ -th algebra by functions labeled such as  $\Psi_i^i$ , satisfying

$$(\Psi_i^i * \Phi_i^i)(x) = \exp \left( \frac{i}{2} \Omega_i^{\mu\nu} \frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x''^\nu} \right) \Psi_i^i(x') \Phi_i^i(x'') \Big|_{x''=x'=x}. \quad (2.17)$$

The reductionist view of what we want to do is to find a way to glue these algebras together cogently into one unified algebra. For that we now return to string theory for intuition.

In string theory the above situation can arise in a configuration of  $N$  D-branes with different but constant  $\mathcal{F}$  on each of them. From the discussion of the last subsection, this can happen in an arbitrary combination of two scenarios. The first, already explained

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<sup>10</sup>  $|x\rangle$  is an eigenstate of  $X_c$  and  $|e_1^a, e_2^a\rangle$  (shorthand for  $|e_1^a\rangle \otimes |e_2^a\rangle$ ) that for  $E_1^a$  and  $E_2^a$ . We have also made a convenient choice for the phase for the basis wave function of these representations.

before, is a background of flat but varying  $B$  field configured in such a way that (only) the pull-back of  $B$  to each D-brane is constant through it. The second scenario is a background gauge field that is constant but breaks the  $U(N)$  gauge symmetry. It is not in general meaningful to talk about constant non-Abelian curvature because it would normally not satisfy the equation of motion or the Bianchi identity, but if all the spatial components of the curvatures are in some Cartan subalgebra than everything is fine. For  $U(N)$  this amounts to being able to diagonalize all spatial components of the curvature as  $N \times N$  matrices. This breaks  $U(N)$  down to  $U(1)^N$  and gives the interpretation of  $N$  D-branes each with a distinct and constant  $U(1)$  background. The  $(i, i)$  strings on each of the D-branes are now complemented by  $(i, j)$  strings, which start on the  $i$ -th brane and end on the  $j$ -th D-brane<sup>11</sup>.

Consider wave functions  $\Psi_j^i$  in the lattice string quantum mechanics approximating to the  $(i, j)$  string. The Hilbert space is a tensor product of  $\mathcal{H}_i \otimes \mathcal{H}_j^*$  and the product rule of the whole algebra is generated by

$$(|\alpha\rangle_i \otimes \langle\beta|_j) * (|\theta\rangle_j \otimes \langle\rho|_k) = (\langle\beta|_j |\theta\rangle_j) (|\alpha\rangle_i \otimes \langle\rho|_k). \quad (2.18)$$

Written in terms of matrix valued functions  $\Psi$  and  $\Phi$  on  $R^D$ , this is

$$(\Psi \times \Phi)_k^i(x) \equiv \sum_j \left( \Psi_j^i *_{ijk} \Phi_k^j \right) (x). \quad (2.19)$$

Note that in (eq. 2.18) the product seems to depend only on  $j$ , but one has to write the final form (eq. 2.19) in the  $aA$  representation. In general that would mean  $*_{ijk}$  depends on all three indices. Our goal is to calculate  $*_{ijk}$ .

### *Preparation and Notations*

Again each brane is labeled by index  $i, j, \dots$  let us denote the  $\mathcal{F}$  on the  $i$ -th D-brane by  $\mathcal{F}^i$ . One repeats the same procedure of constrained quantization. This time one finds the Poisson brackets of the constraint  $D$ 's are

$$\begin{aligned} \{D_\mu^1, D_\nu^1\} &= -4(2\pi\alpha')^2 [\mathcal{F}_i (1 - (2\pi\alpha')^2 \mathcal{F}_i G^{-1} \mathcal{F}_i G^{-1})]_{\mu\nu}, \\ \{D_\mu^2, D_\nu^2\} &= 4(2\pi\alpha')^2 [\mathcal{F}_j (1 - (2\pi\alpha')^2 \mathcal{F}_j G^{-1} \mathcal{F}_j G^{-1})]_{\mu\nu}, \\ \{D_\mu^1, D_\nu^2\} &= 2(2\pi\alpha')^2 [\mathcal{F}_i - \mathcal{F}_j]_{\mu\nu}. \end{aligned} \quad (2.20)$$

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<sup>11</sup> In [3] a system of D0-D4 was studied with the 0-4 strings having mixed boundary conditions. These strings complement the 0-0 strings and the 4-4 strings to produce a bigger algebra.

For  $i \neq j$ ,  $D^1$  and  $D^2$  no longer commute. This would translate to  $X_1$  and  $X_2$  not commuting with each other and would impede the program we have developed for constructing the product. However, we can take the zero slope limit employed in [3], in which  $\alpha' \rightarrow 0$  while  $\mathcal{F}$  and  $(2\pi\alpha')^2 G^{-1}$  remain finite.

After taking the limit, one finds that

$$\begin{aligned} [X_1^\mu, X_1^\nu] &= -i\Omega_i^{\mu\nu}, \\ [X_2^\mu, X_2^\nu] &= i\Omega_j^{\mu\nu}, \\ [X_1^\mu, X_2^\nu] &= 0, \end{aligned} \tag{2.21}$$

where

$$\Omega_i = (\mathcal{F}^i)^{-1}. \tag{2.22}$$

We can always, through a congruence transformation, turn an  $\mathcal{F}$  into the following canonical form:

$$\Omega_i = T_i J T_i^\top, \tag{2.23}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{2.24}$$

It shall become convenient to use the following symbols:

$$\begin{aligned} (1) \quad U_{ij} &\equiv T_i T_j^{-1}; \\ (2) \quad M^{j_1 j_2} &\equiv T_{j_1}^{-1} + T_{j_2}^{-1}; \\ (3) \quad \mathcal{F}^{j_1 j_2; j_3 j_4} &\equiv -\frac{(M^{j_1 j_2})^\top J M^{j_3 j_4}}{4}; \\ (4) \quad \Omega_{j_1 j_2; j_3 j_4} &\equiv (\mathcal{F}^{j_3 j_4; j_1 j_2})^{-1}; \\ (5) \quad S_{j_1 j_2}^{j_3 j_4} &\equiv (M^{j_1 j_2})^{-1} M^{j_3 j_4} \end{aligned} \tag{2.25}$$

They are related to each other and  $\Omega_j$  by

$$\begin{aligned} (1) \quad \mathcal{F}^{j_1 j_2; j_3 j_4}^\top &= -\mathcal{F}^{j_3 j_4; j_1 j_2}; \\ (2) \quad \mathcal{F}^j &= \mathcal{F}^{jj; jj}; \\ (3) \quad \Omega_j &= \Omega_{jj; jj}; \\ (4) \quad S_{j_1 j_2}^{j_3 j_4} &= \Omega_{j_1 j_2; k_1 k_2} \mathcal{F}^{k_1 k_2; j_3 j_4} = S_{j_1 j_2}^{k_1 k_2} S_{k_1 k_2}^{j_3 j_4}; \\ (5) \quad S_{j_1 j_2}^{j_3 j_4} \Omega_{j_3 j_4; j_5 j_6} &= \Omega_{j_1 j_2; j_5 j_6} = \Omega_{j_1 j_2; j_3 j_4} S_{j_5 j_6}^{j_3 j_4}^\top. \end{aligned} \tag{2.26}$$

Note that because we will deal with a plethora of indices we shall suppress spatial indices unless doing so will cause confusion. Repeated gauge indices  $i, j$  are not summed over

unless stated otherwise explicitly, but repeated spatial indices are always summed over implicitly. The situation should be obvious from the context. Coordinates are arranged into column vector, or row vectors after transposition.

### *In search of a center*

In this subsection we figure out the geometric observables for the  $(i, j)$  dipole. That is, we want operators  $X_c^\mu$  such that

$$[X_c^\mu, P_\nu] = \imath \delta_\nu^\mu, \quad (2.27)$$

where the translation generator  $P_\mu$  for the dipole system described by (eq. 2.4) is

$$P = \mathcal{F}^i X_1 - \mathcal{F}^j X_2 \quad (2.28)$$

in the zero slope limit taken earlier.  $P$  satisfies the property

$$[P_\mu, P_\nu] = -\imath(\mathcal{F}_{\mu\nu}^i - \mathcal{F}_{\mu\nu}^j) \equiv -\imath \Delta_{\mu\nu}. \quad (2.29)$$

Therefore they are like covariant derivatives and we require them to be as such:

$$P \equiv \Pi + \tilde{A}, \quad (2.30)$$

where

$$\Pi_\mu = -\imath \frac{\partial}{\partial X_c^\mu}, \quad (2.31)$$

and

$$\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \Delta_{\mu\nu}. \quad (2.32)$$

The definition of  $\tilde{A}$  suffers the usual phase ambiguity and we choose a *linear* gauge

$$\tilde{A} = -\frac{1}{2}(\Delta + \Theta)X_c, \quad (2.33)$$

where  $\Theta$  is a symmetric matrix and pure gauge. It will be fixed later for convenience.

There are an infinite number of choices for operators satisfying (eq. 2.27). Let us for now look for one as close to the center of mass

$$X_{1/2} = \frac{1}{2}(X_1 + X_2) \quad (2.34)$$

as possible. Alas

$$[X_{1/2}^\mu, X_{1/2}^\nu] = -\frac{\imath}{4}(\Omega_i^{\mu\nu} - \Omega_j^{\mu\nu}) \equiv \imath \nabla^{\mu\nu}, \quad (2.35)$$

so  $X_{1/2}$  itself does not suffice. Therefore we define  $X_c$  indirectly so that

$$X_{1/2} = \Lambda X_c + \Gamma \Pi. \quad (2.36)$$

Then  $\Lambda$  and  $\Gamma$  can be found by substituting (eq. 2.31) and (eq. 2.36) into the commutation relation known for  $X_{1/2}$  and  $P$ :

$$\imath \delta_\nu^\mu = [X_{1/2}^\mu, P_\nu] = \imath \left( \Lambda - \frac{1}{2} \Gamma (\Delta - \Theta) \right) \quad (2.37)$$

and

$$\imath \nabla^{\mu\nu} = [X_{1/2}^\mu, X_{1/2}^\nu] = \imath (\Lambda \Gamma^\top - \Gamma \Lambda^\top). \quad (2.38)$$

It thus follows that  $\Lambda$  is related to  $\Gamma$  by

$$\Lambda = \mathbb{I} + \frac{1}{2} \Gamma (\Delta - \Theta) \quad (2.39)$$

So

$$\Gamma^\top - \Gamma + \Gamma \Delta \Gamma^\top = \nabla. \quad (2.40)$$

$\Gamma$  is thus related to a matrix  $\gamma$  satisfying

$$\gamma \Delta \gamma^\top = \Delta - \Delta \nabla \Delta \quad (2.41)$$

through

$$\gamma = 1 + \Delta \Gamma. \quad (2.42)$$

One can then solve for  $X_c$  and find that

$$X_c = (1 + \Gamma \Delta)^{-1} (X_{1/2} - \Gamma P) = (1 + \Gamma \Delta)^{-1} \left( \left( \frac{1}{2} + \Gamma \mathcal{F}^i \right) X_1 + \left( \frac{1}{2} - \Gamma \mathcal{F}^j \right) X_2 \right). \quad (2.43)$$

*Solving for  $\Lambda_{ij}$*

The matrix  $U_{ij}$  defined in (eq. 2.25) satisfy

$$\Omega_i = U_{ij} \Omega_j U_{ij}^T. \quad (2.44)$$

as well as the cocycle condition

$$U_{ij} U_{jk} = U_{ik}. \quad (2.45)$$



From the requirement that the  $X_c$ 's commute among themselves it follows that

$$(1 + 2\Gamma\mathcal{F}^i)^{-1}(1 - 2\Gamma\mathcal{F}^j)$$

also satisfy the condition (eq. 2.44) . We use this to find a solution for  $\Gamma_{ij}$  in terms of  $U_{ij}$ :

$$\Gamma = \frac{1}{2} (1 - U_{ij}) (\mathcal{F}^i U_{ij} + \mathcal{F}^j)^{-1} \quad (2.46)$$

so that

$$U_{ij} = (1 + 2\Gamma\mathcal{F}^i)^{-1}(1 - 2\Gamma\mathcal{F}^j). \quad (2.47)$$

Then one can show that

$$\begin{aligned} (1 + \Gamma\Delta)^{-1} \left( \frac{1}{2} + \Gamma\mathcal{F}^i \right) T_i &= (T_i^{-1} + T_j^{-1})^{-1} \\ &= (1 + \Gamma\Delta)^{-1} \left( \frac{1}{2} - \Gamma\mathcal{F}^j \right) T_j = (M^{ij})^{-1}, \end{aligned} \quad (2.48)$$

where  $M^{ij}$  has been defined in (eq. 2.25) . This allows us to find a simple expression for  $X_c$  that we will use shortly:

$$M^{ij} X_c \equiv (E_1 + E_2), \quad (2.49)$$

where

$$E_1 = T_i^{-1} X_1, \quad E_2 = T_j^{-1} X_2. \quad (2.50)$$

The  $E$ 's are convenient because

$$\begin{aligned} [E_2^\mu, E_2^\nu] &= J^{\mu\nu} = -[E_1^\mu, E_1^\nu], \\ [E_1^\mu, E_2^\nu] &= 0 \end{aligned} \quad (2.51)$$

For  $X_c$  to be defined,  $M$  has to be nondegenerate, which means  $U_{ij}$  has no eigenvalue equal to  $(-1)$ . Actually it might very well happen that for certain given  $\mathcal{F}_i$ 's, for instance the  $SU(2)$  case that we shall consider later,  $U_{ij}$  does have eigenvalue to  $(-1)$ . However,  $U_{ij}$  is only defined up to  $U_{ij} \rightarrow T_i S_i (T_j S_j)^{-1}$ , where  $S_i$  and  $S_j$  are  $Sp(D)$  transformations. It is easy to show that one can always find suitable  $S$ 's so that  $M$  is nondegenerate.

### *The product*

In computing the actual product, the key step is the change of basis functions between the  $aA$  and  $aa$  representations. We now find them for the  $(i, j)$  string. We choose to diagonalize  $E_1^a$  and  $E_2^a$  in the  $aa$  representation,  $a$  ranging from 1 to  $\frac{D}{2}$ . The first  $\frac{D}{2}$  components of  $\frac{1}{2} M^{ij} X_c$  is  $\frac{1}{2} (E_1^a + E_2^a)$ . The canonical conjugates of the rest  $\frac{D}{2}$  components

are  $(E_1^a - E_2^a)$ . To find how the latter are represented, substituting the expressions for  $\Delta$  and  $X_c$  in (eq. 2.30) , we find that

$$\begin{aligned}\Pi &= P + \frac{1}{2}(\Delta + \Theta) X_c \\ &= -\frac{M^\top J}{2}(E_1 - E_2) \\ &\quad + \left(\Theta + T_i^{-1\top} J T_j^{-1} - T_j^{-1\top} J T_i^{-1}\right) (2M)^{-1} (E_1 + E_2)\end{aligned}\tag{2.52}$$

Now we fix the gauge choice by requiring the second term in the last expression vanish. Thus  $(E_1 - E_2)^a$  are represented purely as derivatives with respect to  $(Mx)^A$ . Therefore the change of basis matrix element between the  $(e_1^a, e_2^a)$  basis and the  $X_c$  basis is

$$\langle e_1^a, e_2^a || x \rangle = \frac{\sqrt{|M|}}{2^{\frac{D}{2}}} \exp \left( -\imath (e_1^a - e_2^a) J^{aA} \left( \frac{1}{2} M^{ij} x \right)^A \right) \delta \left( \frac{e_1^a + e_2^a}{2} - \frac{(M^{ij} x)^a}{2} \right). \tag{2.53}$$

The determinant and powers of two appears as a result of the different normalization between  $X_c$  basis and the  $\frac{MX_c}{2}$  basis. They will not matter in the end.

Now we are finally ready to compute the star product.

$$\begin{aligned}(\Psi_j^i * \Phi_k^j)(x) &\propto \int de_1^a de_2^a \langle x || e_1^a, e_2^a \rangle (\Psi_j^i * \Phi_k^j)(e_1^a, e_2^a) \\ &= \int dx' dx'' \int de_1^a de_2^a dm_1^a dm_2^a \Psi_j^i(x') \Phi_k^j(x'') \delta(m_1^a - m_2^a) \\ &\quad \langle x || e_1^a, e_2^a \rangle \langle e_1^a, m_2^a || x' \rangle \langle m_1^a, e_2^a || x'' \rangle \\ &= \frac{\sqrt{|M^{ij} M^{jk} M^{ik}|}}{2^{\frac{D}{2}}} \int dx' dx'' \Psi_j^i(x') \Phi_k^j(x'') \\ &\quad \exp \left( \frac{\imath}{2} (x^\top (\mathcal{F}^{ik;ij} x' - \mathcal{F}^{ik;jk} x'') + x'^\top \mathcal{F}^{ij;jk} x'') \right) \\ &= 2^{\frac{D}{2}} \sqrt{\frac{|M^{ik}|}{|M^{ij} M^{jk}|}} \exp \left( \frac{\imath}{2} \frac{\partial}{\partial x'} \Omega_{ij;jk} \frac{\partial}{\partial x''} \right) \Psi_j^i(x') \Phi_k^j(x'') \Bigg|_{x''=S_{jk}^{ik} x}^{x'=S_{ij}^{ik} x}.\end{aligned}\tag{2.54}$$

Again we fix the normalization by requiring the recovery of the usual matrix product when all the  $\Omega_i$ 's vanish. Hence we would get

$$(\Psi_j^i * \Phi_k^j)(x) = \exp \left( \frac{\imath}{2} \frac{\partial}{\partial x'^\mu} \Omega_{ij;jk}^{\mu\nu} \frac{\partial}{\partial x''^\nu} \right) \Psi_j^i(x') \Phi_k^j(x'') \Bigg|_{x''=S_{jk}^{ik} x}^{x'=S_{ij}^{ik} x}.\tag{2.55}$$

For plane waves, this translate to

$$\exp(\imath k_1 x) *_{ijk} \exp(\imath k_2 x) = \exp \left( -\frac{\imath}{2} k_1^\top \Omega_{ij;jk} k_2 \right) \exp \left( \imath (k_1^\top S_{ij}^{ik} + k_2^\top S_{jk}^{ik}) x \right), \tag{2.56}$$

which is the desired product.

By using (eq. 2.26) , one can show that

$$\begin{aligned}
& (\exp(\imath k_1 x) *_{ijk} \exp(\imath k_2 x)) *_{ikl} \exp(\imath k_3 x) = \exp(\imath k_1 x) *_{ijl} (\exp(\imath k_2 x) *_{jkl} \exp(\imath k_3 x)), \\
& = \exp\left(-\frac{1}{2}\imath(k_1^\top \Omega_{ij;jk} k_2 + k_2^\top \Omega_{jk;kl} k_3 + k_1^\top \Omega_{ij;kl} k_3)\right) \exp(\imath(k_1^\top S_{ij}^{il} + k_2^\top S_{jk}^{il} + k_3^\top S_{kl}^{il})x)
\end{aligned} \tag{2.57}$$

thus proving associativity.

### 3. The Case of $SU(2)$

In this section we deal with the simplest instance of non-Abelian geometry:  $N = 2$  and the deformation parameter is in the adjoint of  $SU(2)$ . That is to say the noncommutativity parameter on brane 1 is  $\Omega$  but that on brane 2 is  $-\Omega$ . For this situation one may certainly apply the method developed in the previous section again. But we shall take this opportunity to consider a variation and illustrate the meaning of the large degree of freedom in choosing the geometric observables mentioned earlier.

For simplification of notation, we can, by means of a congruence transformation, bring  $\Omega$  to its canonical form  $J$  and shall work in this basis till near the end of this section. We call the coordinate observables on the left and right ends of the string  $L^\mu$  and  $R^\mu$  respectively. Unlike the previous section, where  $E_2$  is generically in a different parameterization of  $R^D$  from  $E_1$ , related by some linear transformation, here  $R$  is the same parameterization as  $L$ . Therefore  $[L^\mu, L^\nu]$  and  $[R^\mu, R^\nu]$  are exactly opposite in sign on a 11 string, but identical on a 12 string.

#### 3.1. Split Ordering

The Moyal-Weyl product can serve as a method of quantization, i.e. mapping a function on the phase space (in our case,  $R^D$ ) to an operator to the Hilbert space of a quantum mechanical model. As usual there is the ambiguity of operator ordering, and Moyal-Weyl product makes a symmetric choice. There are other orderings, and they can also be obtained by variation of the method developed in the [5] and reviewed in the last section. Recall that to represent states in the LSQM Hilbert space as (wave) functions on  $R^D$ , we had to choose a set of  $D$  geometric observables, simultaneously diagonalized in the  $aA$  representation. The action on them by the generator of translation should be what one expects for coordinates being translated. In the last section,  $X_c$ 's are the geometric observables, but there are many other choices. Some of them, giving different values to  $\Theta$ , correspond to different choices of phase for the wave function. Some other choices

corresponds to different operator ordering schemes in the language of quantization. Both will show up here.

Let us divide the coordinates of the present problem into two groups which are canonical conjugates to each other with respect to  $J$  (and  $-J$ ). We label them with  $a, b, \dots$  and  $A, B, \dots$  respectively as in section 2.1. Then we choose as geometric observables for any  $(i, j)$  string

$$\begin{aligned} E^a &= L^a, \\ E^A &= R^A. \end{aligned} \tag{3.1}$$

Now let us consider the 11 string. First we will describe a scheme for illustration only that will not be used again in the paper. Therefore to avoid confusion we use  $\doteq$  instead of  $=$  in equations peculiar to this example. The basic commutation relations are

$$\begin{aligned} -[L^\mu, L^\nu] &= J^{\mu\nu} = [R^\mu, R^\nu], \\ [L^\mu, R^\nu] &= 0. \end{aligned} \tag{3.2}$$

The translation operator is

$$P = -J(L - R). \tag{3.3}$$

By a specific choice of phase of the basis state in the  $aA$  representation, we can implement translation by differentiation with respect to the space coordinates as per tradition:  $P = \Pi$ . This means in particular that

$$R^a \doteq -iJ^{aA} \frac{\partial}{\partial e^A} + e^a. \tag{3.4}$$

Then by another choice of phase the change of basis between  $aA$  and  $aa$  basis is described by

$$\langle e || L^a, R^a \rangle \doteq \delta(e^a - L^a) \exp^{i(R^a - L^a)J^{aA}e^A}. \tag{3.5}$$

Then one finds that the noncommutative product is given by

$$\Psi_1^1 * \Phi_1^1(e) \doteq e^{i \frac{\partial}{\partial M^A} J^{Aa} \frac{\partial}{\partial M^a}} \Psi_1^1(e^a, M^A) \Phi_1^1(M^a, e^A) \bigg|_{M^a=e^a}^{M^A=e^A}. \tag{3.6}$$

This corresponds, in quantization, to a choice of ordering in which all the  $E^a$ 's are brought to the left and all the  $E^A$ 's are brought to the right.

However, in the rest of the paper we shall only use a variant of this ordering so that the condition  $\alpha_1 = \alpha_2$  is satisfied in (eq. 2.11) and the final result could be in a more

convenient form. Another choice of phase in the  $aA$  representation is made which replaces (eq. 3.4) by

$$R^a = -\imath J^{aA} \frac{\partial}{\partial e^A}. \quad (3.7)$$

Then (eq. 3.5) is replaced by

$$\langle e || L^a, R^a \rangle = \delta(e^a - L^a) \exp(\imath R^a J^{aA} e^A), \quad (3.8)$$

and (eq. 3.6) by

$$\Psi_1^1 * \Phi_1^1(e) = \exp\left(\imath \frac{\partial}{\partial M^A} J^{Aa} \frac{\partial}{\partial M^a}\right) \Psi_1^1(e^a, M^A) \Phi_1^1(M^a, e^A) \Big|_{M^A=0=M^a}. \quad (3.9)$$

The  $U(1)$  phase that relates this and the last one is given by the unit element in this new product. Instead of 1, it is  $\exp(\imath e^a J^{aA} e^A)$ . We call this scheme *split ordering*.

### 3.2. Off diagonal elements

On a 12 string the commutation relations are

$$\begin{aligned} [L^\mu, L^\nu] &= -J^{\mu\nu} = [R^\mu, R^\nu], \\ [L^\mu, R^\nu] &= 0. \end{aligned} \quad (3.10)$$

The translation operator is

$$P = -J(L + R). \quad (3.11)$$

A crucially new feature is that  $P$  no longer commutes among themselves:  $[P^\mu, P^\nu] = -2\imath J^{\mu\nu}$ . By a specific choice of gauge we can implement it as

$$P_\mu = -\imath \frac{\partial}{\partial e^\mu} - [Je]_\mu. \quad (3.12)$$

This means in particular

$$R^a = -\imath J^{aA} \frac{\partial}{\partial e^A}. \quad (3.13)$$

Then by a choice of phase consistent with the split ordering the change of basis between  $aA$  and  $aa$  basis is described by

$$\langle e^a, e^A || L^a, R^a \rangle = \delta(e^a - L^a) \exp(\imath R^a J^{aA} e^A). \quad (3.14)$$

Then one finds that the noncommutative product is given by

$$\Psi_1^1 * \Phi_2^1(e) = \exp \left( i \frac{\partial}{\partial M^A} J^{Aa} \frac{\partial}{\partial M^a} \right) \Psi_1^1(e^a, M^A) \Phi_2^1(M^a, e^A) \Big|_{M^A=0=M^a}. \quad (3.15)$$

Using this method systematically we find all the possible products  $\Psi_j^i *_{ijk} \Phi_k^j$ . They in fact can be written in a very compact matrix form:

$$(\Psi \times \Phi)(e^a, e^A) \equiv \Psi(e^a, M^A) \exp \left( i \frac{\partial}{\partial M^A} \hat{\Omega}^{Aa} \frac{\partial}{\partial M^a} \right) \Phi(M^a, e^A) \Big|_{M^A=0=M^a}, \quad (3.16)$$

where  $\hat{\Omega}^{Aa} = J^{Aa} \sigma_3$  and all products are understood as matrix products<sup>12</sup>. This is highly suggestive of an  $SU(2)$  valued Poisson structure  $\hat{\Omega}$ . For each  $(\mu, \nu)$  pair,  $\hat{\Omega}^{\mu\nu}$  is a two by two matrix, intuitively in the adjoint of  $SU(2)$ . In this case,

$$\hat{\Omega}^{\mu\nu} = J^{\mu\nu} \sigma_3. \quad (3.18)$$

This product is clearly associative.

The unit element of this new product is

$$\mathbb{I}_{su(2)} = \exp \left( -i e^a \hat{\mathcal{F}}_{aA} e^A \right), \quad (3.19)$$

where

$$\hat{\mathcal{F}}_{\mu\nu} = -J^{\mu\nu} \sigma_3. \quad (3.20)$$

### 3.3. Non-Abelian Geometry

Just as in the general case discussed in the last section, constant matrices form a subalgebra. However,

$$\Psi \times M = \Psi(e^a, 0)M \neq M\Psi(0, e^A) = M \times \Psi, \quad (3.21)$$

unless  $\Psi(e^a, 0) = \Psi(0, e^A)$  is a constant matrix that commutes with  $M$ . Therefore one cannot obtain the whole algebra by tensoring this matrix subalgebra with some other

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<sup>12</sup>  $\sigma_3$  is just the usual element of Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.17)$$

algebra. Curiously, there are two other *distinct* matrix subalgebras with the interesting properties:

$$(M\mathbb{I}_{su(2)}) \times \Psi = M\Psi, \quad \Psi \times (\mathbb{I}_{su(2)}M) = \Psi M, \quad (3.22)$$

so that the total algebra is a left and right module under them separately and respectively.

The new algebra defined by (eq. 3.16) and (eq. 2.55) also contains subalgebras that are the deformations of that of the scalar functions on  $R^D$ . However, there are  $N$ , rather than just one, of them, distinguished by their deformation parameters  $\Omega_i$ . No one is more preferred than the others. On the other hand, a well defined deformed algebra of functions is essential for the noncommutative geometric interpretation of D-brane worldvolume. A noncommutative space is itself defined only by the algebra of functions “on it.” The loss of a canonical noncommutative algebra of scalar functions calls for a drastic reinterpretation of the underlying “space.” In the present case, the  $N$  different algebras represent  $N$  deformed noncommutative spaces on top of each other, distinguished only by their deformation parameters. However, this simplistic picture overlooks all the  $(i, j)$  strings. Indeed, it is clearly not “covariant” enough. The total algebra is not a simple tensor product of any one of the scalar subalgebras with some matrix algebra, as it were for the usual case of Abelian or no deformation. What is the meaning of this?

We propose that these algebra define examples of a new type of geometry, which we call *non-Abelian Geometry*. It is a type quite apart from both the original underlying commutative space  $R^D$  and the noncommutative  $R^D$  defined by the Moyal product because the matrix (and more generally, non-Abelian) degree of freedom and the function degree of freedom become entangled everywhere and become one entity. Recall that the algebra of functions of the direct product of two manifolds,  $M \times M'$ , is the tensor product of their respective algebras:

$$A_{M \times M'} = A_M \otimes A_{M'}. \quad (3.23)$$

So should be the case for the direct product of noncommutative spaces. Now also recall that the algebra of  $N \times N$  matrices can be reinterpreted as the algebra of functions on a certain discrete noncommutative space: a discrete “fuzzy” torus with  $N$  units of magnetic flux<sup>13</sup>. Therefore the usual case of  $N \times N$  matrix fields on commutative or  $U(1)$  deformed space can be reinterpreted as (the algebra on) the direct product of the continuous space with the

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<sup>13</sup> This can be constructed, for example, on a two-torus as follows. With  $N$  (more generally, rational) units of flux through the torus, the algebra of functions contains centers. One finds that the algebra has an  $N$  dimensional irreducible representation in which the Fourier components are realized as clock and shift operators of a  $N$ -ary clock and the products thereof. The latter generate the algebra of  $N \times N$  matrices.

appropriate discrete torus. When the  $U(N)$  bundle is nontrivial, it should be identified as a fibration of the discrete torus over the continuous base space. The notion of a base space is based on the existence of a canonical algebra of scalar valued functions on it, of which the total algebra is a module. This semi-decoupling makes it a matter of taste whether to consider the total algebra as representative of a combined noncommutative space or just conventionally as an adjoint module of the algebra of the base space. However, with non-Abelian deformation considered in this paper, such a canonical scalar algebra ceases to exist. The discrete torus and the continuous space therefore lose their independent identities and separate meaning. The conventional picture has to be replaced by a total space that intertwines the discrete, non-Abelian degrees of freedom of the Yang- Mills theory with that of the continuous, noncommutative space. This is the precise meaning of *non-Abelian geometry*.

#### 4. The Taub-NUT connection

So far we have studied an example of Lorentz non-invariant theory. These theories give new deformations to the otherwise constrained structure of quantum field theory. As discussed above, they can be realized in string theory when we have a background B-field. In the presence of branes we have basically four choices of orienting the B-field resulting in four different theories. The first case would be to orient the B-field transverse to the brane [12] i.e the B-field is polarized orthogonally. Naively such a constant B-field can be gauged away. However if we also have a nontrivial orthogonal space – say a Taub-NUT – and one leg of the B-field is along the Taub-NUT cycle then this configuration give rise to new theories known as the pinned brane theories[12]. The D-branes have minimal tension at the origin of the Taub-NUT and therefore the hypermultiplets in these theories are massive. The mass is given by

$$m^2 = \frac{b^2}{1 + b^2} \quad (4.1)$$

where  $b$  is the expectation value of the B-field at infinity. The origin of the mass of the hypermultiplets is easy to see from the T-dual version. For simplicity we will take a D3 brane oriented along  $x^{0,1,2,3}$  and is orthogonal to a Taub-NUT which has a non trivial metric along  $x^{6,7,8,9}$ . The coordinate  $x^6$  is the Taub-NUT cycle and the B-field has polarization  $B_{56}$ . Making a T-duality along the compact direction of the Taub-NUT we have a configuration of a NS5 brane and a D4 brane. The hypermultiplets in this model come from strings on the D4 brane crossing the NS5 brane. Due to the twist on



the torus  $x^{5,6}$ , the D4 on the NS5 brane comes back to itself by a shift resulting in the hypermultiplets being massive while the vectors remain massless.

The second case is to orient the B-field with one leg along the brane and the other leg orthogonal to it [13]. Again we could gauge away such a B-field. But in the presence of Taub-NUT – with the leg of the B-field along the Taub-NUT cycle – we generate new theories on the brane known as the dipole theories[14]. Hypermultiplets in these theories have dipole length  $L$  determined by the expectation of the B-field. The vector multiplets have zero dipole lengths. The dipoles are light and typically the branes are not pinned. The field theory on the branes are nonlocal theories with the following multiplication rule:

$$(\Phi \circ \Psi)(x) \equiv \exp \left( \frac{1}{2} (L_1^i \frac{\partial}{\partial x''^i} - L_2^j \frac{\partial}{\partial x'^j}) \right) \Phi(x') \Psi(x'') \Big|_{x'=x''=x} \quad (4.2)$$

where  $\Phi(x)$  and  $\Psi(x)$  have dipole lengths  $L_1$  and  $L_2$  respectively. It is easy to check that when we specify the dipole length of the above product as  $L_1 + L_2$ , the multiplication rule is associative. The dipoles in these theories are actually rotating arched strings stabilized (at weak coupling) by a generalized magnetic force[13]. In this limit the radiation damping and the coulomb attraction are negligible. Again the T-dual model can illustrate why the hypermultiplets have dipole length. We take the above configuration of a D3 transverse to a Taub-NUT but now with a B-field  $B_{16}$ . Under T-duality we get a configuration of a NS5 brane and a D4 brane with a twisted  $x^{1,6}$  torus. Along direction  $x^1$  the D4 comes back to itself up to a twist. Since both the NS5 and the D4 branes are *along*  $x^1$ , this shift gives a dipole length to the hypermultiplets. Observe that this way the vectors have zero dipole length.

The third case is to orient the B-field completely along the branes. Here we cannot gauge away the B-field. Gauging will give rise to  $F$  field on the world volume of the branes. This would also mean that now we no longer need any nontrivial manifold. The supersymmetry will thus be maximal (in the above two cases the supersymmetry was reduced by half or less). The theory on the brane is noncommutative gauge theory. The B-field modifies the boundary conditions of the open strings describing the D branes. This modification is crucial in giving a non zero correlation function for three (and more) gauge propagators. This in turn tells us that the usual kinetic term of the gauge theory is replaced by<sup>14</sup>

$$\int \sqrt{\det g} g^{ii'} g^{jj'} F_{ij} * F_{i'j'} \quad (4.3)$$

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<sup>14</sup>  $g_{ij}$  is the open string metric  $G_{ij}$  of [3].

where (eq. 4.3) involves an infinite sequence of terms due to the definition of  $*$  product (eq. 2.16). The above equation is written in terms of *local* variables i.e the variables defining the usual commutative Yang-Mills theory. The map which enables us to do this is found in [3]. In other words, noncommutative YM at low energies can be viewed as a simple tensor deformation of commutative YM. This deformation is also responsible in producing a scale  $\Omega$  in the theory. A key feature is that this scale governs the size of smallest lump of energy that can be stored in space. Any lump of size smaller than this will have more energy and therefore will not be physically stable. For a D3 brane with a B-field  $B_{23}$  T-duality along  $x^3$  will give a D2 brane on a twisted  $x^{23}$  torus. The non locality in this picture can be seen from the string which goes around the  $x^3$  circle and reaches the D2 with a shift [2].

For the dipole theories (which are again non-local theories) there also exist a map with which we could write these theories in terms of local variables[14]. This map is relatively simpler than the Seiberg-Witten map for noncommutative theories. Using this map one can show that the dipole theories at low energies are simple vector deformations of SYM theory [13].

The fourth case is the topic of this paper. Here, as discussed earlier, we have a configuration of multiple D-branes with different  $B$ -fields oriented parallel to the branes. However one difference now is that this configuration may or may not preserve any supersymmetry. Also the multiplication rule in this theory is more complex and now there is no clear distinction between the non-Abelian and the noncommutative spaces. The algebra (eq. 2.55) reflects this intertwining.

From the above considerations it would seem that all the four theories have distinct origins. However, as we shall discuss below, all these theories can be derived from a particular setup in M-theory but with different limits of the background parameters. This will give us a unified way to understand many of the properties of these theories.

Consider first the pinned brane case. If we lift a D4 brane with transversely polarized B-field we will have a configuration of a M5 brane near a Taub-NUT singularity and a C-field having one leg parallel to the M5 brane. The limits of the external parameters which give rise to decoupled theory on the M5 brane are[12]:

$$C \rightarrow \epsilon, \quad R \rightarrow \epsilon, \quad M_p \rightarrow \epsilon^{-\beta}, \quad \beta > 1 \quad (4.4)$$

where  $R$  is the Taub-NUT radius and  $M_p$  is the Planck mass. In this limit the energy scale of the excitations of the M5 brane is kept finite whereas the other scales in the problem are set to infinity.

The dipole theories are now easy to get from the above configuration. Keeping the background limits same we rotate the M5 brane such that the C-field now has two legs along the M5 (its still orthogonal to the Taub-NUT). With this choice a simple calculation will tell us that the M5 is not pinned in this case.

To generate the noncommutative gauge theories we first remove the M5 brane from the picture and identify the M-theory direction with the Taub-NUT circle. Now the limits<sup>15</sup> which give rise to  $6 + 1$  dimensional noncommutative gauge theories are[16]:

$$C \rightarrow \epsilon^{-1/2}, \quad R \rightarrow \text{fixed}, \quad M_p \rightarrow \epsilon^{-1/6}, \quad g_{\mu\nu}^M \rightarrow \epsilon^{2/3} \quad (4.5)$$

where  $g_{\mu\nu}^M$  is the dimensionless M-theory metric<sup>16</sup>. This limit is the same as Seiberg-Witten limit and the coupling constant of the theory

$$g_{YM}^2 = M_p^{-3} C = \text{fixed}. \quad (4.6)$$

The theory of non-Abelian geometry can be studied in M-theory using a multi Taub-NUT background with a  $G$ -flux that has non-zero expectation values near the Taub-NUT singularities. As discussed in [17], such a choice of background flux generally breaks supersymmetry. From type IIA D6 brane point of view this flux will appear as gauge fluxes  $F_i = dA_i$  on the  $i^{th}$  world volume<sup>17</sup>. If the Taub-NUT is oriented along  $x^{7,8,9,10}$ ,  $x^7$  being the Taub-NUT circle, and the  $C$  field has two legs along the Taub-NUT and one leg along  $x^1$  then the world volume gauge fields  $A_i$  can be determined by decomposing the  $C$  field as:

$$C(t, y, x^7, \vec{r}) = \sum_{i=1}^N A_i(t, y) \wedge \mathcal{L}_i^{(2)}(x^7, \vec{r}) \quad (4.7)$$

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<sup>15</sup> These limits however don't specify the complete decoupling of the theory. This is because the theory has negative specific heat[15].

<sup>16</sup> There is an interesting digression to the above cases. Between the two limits of the background  $C$  field there exist a case under which

$$C \rightarrow \text{finite}, \quad R \rightarrow \text{finite}, \quad M_p \rightarrow \infty, \quad g_s \rightarrow 0$$

This gives us another decoupled theory on the M5 brane which is in the same spirit as the little string theory[12].

<sup>17</sup> There is a subtlety here. For F/M-theory compactification with  $G$ -flux, when there is a generic flux— not concentrated near the singularities of the manifold — this appears in the corresponding type IIB theory as  $H_{NSNS}$  and  $H_{RR}$  background. However when the flux is concentrated near the singularity, then it appears as gauge fields on the brane[17].

where  $y$ 's are the coordinates  $x^{1,\dots,6}$  of the  $D6$  brane world volume,  $\mathcal{L}_i^{(2)}$  are the harmonic forms of the multi Taub-NUT and  $|\vec{r}| = \sqrt{x^k x^k}$ ,  $k = 8, 9, 10$ . However it turns out that these harmonic forms are *deformed* from their original values due to the background  $G$ -flux. Therefore in the above equation the harmonic forms are  $C$ -twisted ones which preserve their anti-self-duality with respect to the  $C$ -twisted background metric. The precise form of this twist will be presented elsewhere [18]. In this frame-work it might be possible – by pure geometric means – to see this intertwining more clearly.

Before we end this section let us summarize the connections between different theories governed by the  $B$  field dynamics in the following table<sup>18</sup>:

<i>Theories</i>	<i>SUSY</i>	<i>Product rule: <math>(\Phi_j^i * \Psi_k^j)(x)</math></i>
Pinned branes	$\mathcal{N} = 2$	$\Phi_j^i(x) \Psi_k^j(x)$
Dipole theory	$\mathcal{N} \leq 2$	$e^{\frac{1}{2}(L_1^i \frac{\partial}{\partial x'^i} - L_2^j \frac{\partial}{\partial x'^j})} \Phi_j^i(x') \Psi_k^j(x'') _{x'=x''=x}$
Noncomm. geometry	$\mathcal{N} = 4$	$e^{(\frac{1}{2} \frac{\partial}{\partial x'^\mu} \Omega^{\mu\nu} \frac{\partial}{\partial x''^\nu})} \Phi_j^i(x') \Psi_k^j(x'') _{x''=x'=x}$
Non-Abelian geometry	$\mathcal{N} \leq 1$	$e^{(\frac{1}{2} \frac{\partial}{\partial x'^\mu} \Omega_{ij;jk}^{\mu\nu} \frac{\partial}{\partial x''^\nu})} \Phi_j^i(x') \Psi_k^j(x'') _{\substack{x'=S_{ij}^{ik}x \\ x''=S_{jk}^{ik}x}}$

## 5. Discussions and Conclusion

In this section we will discuss possible supergravity background for the analysis presented in the earlier sections. We will also illustrate some aspects of this using a mode expansion for a background  $U(1) \times U(1)$ . This case is related to the recent analysis done in [19], where the worldsheet propagator was calculated to compute two distinct noncommutativity parameter. It was shown that near one of the brane, say 1, the  $*$ -product involves  $\Omega_1$  only. This is clear from our analysis because  $\Psi_1^1 * \Phi_1^1$  involves  $\Omega_{11;11}$  which from (eq. 2.26) is just  $\Omega_1$ .

Another recent paper which dealt with some related aspects is [9]. Here two different  $*$  products arise naturally in the fractional brane setting. In this model we have a configuration of  $D5 - \overline{D5}$  wrapping a vanishing two cycle of a Calabi-Yau. This model however is supersymmetric and for some special choice of background  $B$  and  $F$  fields the tachyon is massless[9].

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<sup>18</sup> We have chosen to make everything non-Abelian. Its an straightforward exercise to extend the above three Abelian cases to this.

### 5.1. Gravity Solutions

Let us first consider the case of a large number of D3 branes on top of each other and with a background B-field switched on. The B-field is constant along the world volume of the D3 branes. What is the supergravity solution for the system? Obviously the near horizon geometry cannot be AdS as there is a scale  $\Omega$  in the theory which breaks the conformal invariance. Indeed, as shown by Hashimoto and Itzhaki[20] and independently by Maldacena and Russo[15], the supergravity solution can be calculated by making a simple T-duality of the D3 brane solution. Under a T-duality the B background becomes metric and it tilts the torus which the D2 brane wraps. This solution is known and therefore we could calculate the metric for this case and T-dualize to get our required solution. Observe that under a T-duality we do get a B-field which is constant *along* the brane but is a nontrivial function along the directions orthogonal to the brane. In other words there is a  $H$  field. But for all practical purposes this solution is good enough to give us the near horizon geometry of the system. The scale of the theory appears in the metric deforming our AdS background which one would expect in the absence of B-field. For the case in which D3 branes are along  $x^0, x^1, \dots, x^3$  and B-field has a polarization  $B_{23}$  the near horizon geometry looks like[20,15]:

$$ds^2 = \alpha' \left[ u^2 (-dx_0^2 + dx_1^2) + u^2 h (dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right] \quad (5.1)$$

where  $h = (1 + a^4 u^4)^{-1}$  and  $a^2$  is the typical scale in the theory (it is related to  $\theta$ ).

The above metric has the expected behavior that for small  $u$  the theory reduces to  $AdS_5 \times S^5$ . From gauge theory this is the IR regime of the theory. One naturally expects that noncommutative YM reduces to ordinary YM at large distances. The above solution has an added advantage that it tells us that below the scale  $a$  (which will be proportional to  $\sqrt{\Omega}$ ) commutative variables are no longer the right parameters to describe the system accurately. Noncommutativity becomes the inherent property of the system and therefore local variables fail to capture all the dynamics.

At this point we should ask whether our new system has a consistent large  $N$  behavior. The gauge theory is highly noncommutative of course but it also has a large number of noncommutativity parameters (typically  $N$ ). We have a system of  $N$  D3 branes with gauge fields  $F^i, i = 1, \dots, N$  on them. We can simplify the problem by taking only one polarization of the gauge fields, i.e we would concentrate on the fields  $F_{23}^i$ .

A simple analysis tells us immediately that the previous procedure to generate a solution is not going to help in this case. The procedure is suitable to generate *one* scale

and therefore we should now rely on different technique. Also the system now has no supersymmetry and therefore we have to carefully interpret the background.

Let us denote the magnetic field  $F_{23}^i$  on the branes as  $\mathcal{B}_i$ . We can make a Lorentz transformation to generate a *constant* magnetic field  $\mathcal{B}$  on the branes but different electric fields  $\mathcal{E}_i$  such that the relations

$$\mathcal{B}_i^2 = \mathcal{B}^2 - \mathcal{E}_i^2, \quad \mathcal{E}_i \bullet B = 0$$

are satisfied. We can also make a gauge transformation to convert the constant magnetic field to a background constant B-field.

We now make a T-duality along the  $x^3$  direction<sup>19</sup>. Under this the electric fields  $\mathcal{E}_i$  will become velocities  $v_i$  of the D2 branes and the B-field will tilt the torus  $x^2, x^3$  as before. therefore the final configuration will be a bunch of D2 branes (or, in a reduced sense, points) moving with velocities  $v_i$  along the circle  $x^3$ .

At this point it would seem that the supergravity solution is easy to write down. But there are some subtleties here. Recall that when we had a single scale  $\Omega$  in the problem and the T-dual picture was a D2 brane wrapped on a tilted torus, T-duality along  $x^3$  was easy because we had assumed that the harmonic function of the D2 brane is *delocalized* along the third direction. Therefore the D2 brane is actually *smeared* along that direction. This typically has the effect that the harmonic function of the D2 brane is no longer  $1 + \frac{Q_2}{r^5}$  rather its  $1 + \frac{Q_2}{r^4}$ ,  $Q_2$  is the charge of the D2 brane. This is the case that we have to consider. Delocalizing the D2 branes would mean that we have an infinite array of D2 branes moving with velocity, say,  $v_1$  and so on. Also since the system lacks supersymmetry the velocities are *not* constant. An interpretation of this model can be given from fluid mechanics. Due to delocalization we have layers of fluid moving with velocities  $v_i$  along  $x^3$  with a viscosity between them. This would tend to retard the motion of the various layers making the problem slightly nontrivial. But as we shall see some interesting property of the system is obvious without going to the original (T-dual) model.

The metric for the D2 brane (for simplicity the direction 23 are on a square torus) is given by

$$ds^2 = H^{-1/2} ds_{012}^2 + H^{1/2} ds_{34..9}^2 \quad (5.2)$$

where the harmonic function satisfy:

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<sup>19</sup> There could be a subtlety in performing a T-duality here because the string theory background is not supersymmetric. But we are considering a T-duality completely from the supergravity point of view in which the transformation of the bosonic background is important for us. As such the extra corrections are not relevant for studying this.

$$\partial^2 H = \sum_{i=1}^N \delta(r_i)$$

and  $r_i$  is given by  $r_i = \sqrt{(x_3 - v_i t)^2 + x_4^2 + \dots + x_9^2}$  when the velocities are small so that we could neglect relativistic effects. Recall that the system is delocalized along direction  $x_3$  therefore there are actually an infinite array of branes moving (i.e it behaves like a fluid).

Let  $r = \sqrt{x_4^2 + \dots + x_9^2}$  then its easy to show that for a large radius of  $x_3$  circle and near horizon geometry (i.e.  $r \rightarrow 0$ ) the harmonic function is modified from the naive expected value. The harmonic function becomes:

$$H(r) = 1 + \sum_{i=1}^N \frac{1}{r^4} \int_0^{\pi/2} \frac{\sin^3 \theta}{(1 - \frac{v_i t}{r} \sin \theta)^5} d\theta \quad (5.3)$$

when the compact direction is very small one can show that we get  $H(r) = 1 + \frac{1}{r^4}$ . A T-duality along that direction will give us a noncompact D3 brane whose harmonic function will have the right property. This calculation is done without assuming any force between the branes. A more detailed analysis would require the behavior of open strings between the branes. In the next section we will elaborate on this issue by doing a mode expansion.

## 5.2. Mode Expansion

For simplicity we will take two D3 branes having fluxes  $F_i = F_{23}^{(i)}$  on them. The D3 branes are oriented along  $x^{0,1,2,3}$  and let  $z = x^2 + ix^3$  such that the mode expansion for the system becomes:

$$z = \sum_n A_{n+\nu} \exp((n+\nu)t) \left[ \exp(i(n+\nu)\sigma) + \left( \frac{1-iF_1}{1+iF_1} \right) \exp(-i(n+\nu)\sigma) \right] \quad (5.4)$$

The quantity  $\nu$  measures the shift in the mode number due to the presence of different gauge fluxes at the boundary. This shift can be easily worked out from the boundary conditions at the two ends of the open string. In terms of the above variables  $\nu$  is given by

$$\nu = \frac{1}{2\pi} \sin^{-1} \left[ 2 \frac{(F_2 - F_1)(1 + F_1 F_2)}{(1 + F_1^2)(1 + F_2^2)} \right] \quad (5.5)$$

As is obvious from the above formula when the gauge fluxes are same on the different branes we do not expect any shift in the mode number. This shift can now be used to calculate the new ground state energy of the system. This zero-point energy, in the NS sector, will now depend on  $\nu$ . Using the identity

$$\sum_{n \geq 0} (n + \nu) = -\frac{1}{12} (6\nu^2 - 6\nu + 1) \quad (5.6)$$

the zero-point energy can be calculated from the bosons, fermions and the ghosts contributions. The bosons and the fermions along directions  $x^{2,3}$  are quantized with mode numbers  $n + \nu$  and  $n \pm |\nu - \frac{1}{2}|$  respectively. In general for a system of  $Dp$  branes with fluxes  $F_{1,2}$  the zero point energy is given by

$$-\frac{1}{2\alpha'} \left( \frac{p-1}{4} + \left| \nu - \frac{1}{2} \right| \right) \quad (5.7)$$

From the above formula its clear that there is a tachyon on D3 brane for any values of  $\nu$ . For very small values of  $\nu$  the tachyon has  $m^2 = -\frac{1}{2\alpha'}(1 - \nu)$  and for large values of  $\nu$  it has  $m^2 = -\frac{\nu}{2\alpha'}$ . Also now there is a subtlety about GSO projection. Therefore it depends whether we study a  $D1$  brane or  $\overline{D1}$  brane. When the branes are kept far apart then there would be no tachyon in the system but the branes will be attracted to each other which in turn will retard the velocities of the brane.

Let us now consider the special case of  $SU(2)$ . For this we have the following background

$$F_1 = -F_2 = -F \quad (5.8)$$

It is straightforward to show that now the modes will be shifted by  $\nu$  given as

$$\nu = \frac{2}{\pi} \tan^{-1} F \quad (5.9)$$

For the case we are interested in,  $F \rightarrow \infty$ , and therefore the shift  $\nu = 1$ . The ground state energy do not change but all the modes of the string get shifted by 1.

It would be worthwhile to analyze in greater detail the spectrum and dynamics of this theory.

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